When in Rome do as the Romans do: When solving an economist's problem solve the economist's problem: a critique on the criticism of Daniel Kahneman's of Daniel Bernoulli's utility solution to the St Petersburg Paradox

(Work in progress)

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Abstract:

Economics is a multidisciplinary subject and as such attracts academics from a wide variety of disciplines. Many of whom have made considerable contributions to economics. Examples include the mathematician, John von Neumann of game theory fame and more recently the psychologists Amos Tversky and the Nobel Laureate Daniel Kahneman. The ostensible purpose of venturing, straying, out of their respective fields into economics is to assist and indeed to solve the problems economists have been working on. Unfortunately, the staying academic may not clearly understand the economist's problem or the progress in solving the problem and as a consequence the economist's proposed solution. Then while under the impression of solving the economist's problem the straying academic is in fact posing his own problem and proposes a solution to that problem from within his own discipline. In a sense the economist's problem is hijacked and a totally irrelevant solution (irrelevant to the economist's problem that is) is put forward as a proposed solution to solve a different problem. Even worse without understanding the economist problem or solution the straying academic criticises the seminal work of the economist; alleging the economist got it all wrong, without really understanding the economist's problem or the proposes solution. This paper examines an example of this in the criticism of Daniel Kahneman's of Daniel Bernoulli's solution to the St Petersburg paradox and argue that Kahnman's criticism of "Bernoulli's Error" is misplaced. This does not underplay Kahneman's considerable contribution to economics but it is independently so; it does not assist in solving the problem Bernoulli was working on and failed to solve. The paper will argue not much wrong with Daniel Bernoulli's formulation of the problem or his proposed solution as so formulated, for what it is worth. What is wrong with

Kahneman's criticism is he did not understand Bernoulli's problem and does not provide any meaningful error in his "Bernoulli's error".

Introduction

The age when a single individual could master the entire body of knowledge or a substantial part of it has long passed. To-day the body of knowledge is dispersed within various academic disciplines. The body of knowledge is thus not a continent but more akin to an archipelago. One can thus imagine there is an island called mathematics and another called psychology and of course there is an island called economics Some islands are quite big and have counties. The island of economics has a country called macro-economics and another called micro-economics and so on. The islands are filled with industrious people all busy trying to solve problems and thereby adding to the body of knowledge. Occasionally people from one island visit other island and may become interested in contributing to solving problems on the island they are visiting. So for example a person from the island called mathematics may visit the island of economics and become engrossed in the problems of that island and contribute to the solution of the problem the economist is working on. The new insights the visitor brings to bear on the problem may well be invaluable and make a major contribution but on the other hand the visitor may well bring fresh challengers. There is a saying that if the only tool one has is a hammer then every problem looks like a nail. The insights the visitor brings his well be a mere reflection of what he has learnt on his own island and may not be a good solution to the problem, or a solution at all. He may well not even have applied his mind to the problem or even attempted to understand it. The problem and solution then becomes one of confustion and may detract to a solution to the original problem ever being found. It is at this point another adage throws light on the problem; which says, "When in Rome, do as the Romans do". In the context of the economists' island and economist's problem one may adapt the adage and to say "When solving the economist's problem, solve the economist's problem" When proposing a solution to a problem an economist is dealing with make sure it is indeed the problem the economist is trying to solve. The visitor may well be making a substantial contribution to a problem but it is a different problem but not the one the economist was trying to solve. The visitor may well go home one day taking his problem to his island and thus variations, to different problems and different solutions may begin to appear on different islands. In the end this process may not be as helpful as just solving the original problem which may lay unsolved, unnoticed for centuries. The same goes for the economist from one county visiting another county. The economist should first make sure he or she understands the problem his economist colleague was working on and not muddy the waters with confused solutions; indeed non-solutions. The solution put forward may well be a non-solution although in some or other sense important.

Now this may well sound very amusing so I will apply this to a real problem, namely Daniel Kahneman's (psychologist) criticism of Daniel Bernoulli's St Petersburg non-solution to St Petersburg game which led his St Petersburg paradox. Having done so I will look at Daniel Kahneman's strange criticism of Daniel Bernoulli's non-solution. This 300 year confusion would have been avoided if only contributors had concentrated on the original problem.

The problem

Now so as not, for ourselves, to fall into the trap just identified, it is important to first identify the problem for which a solution was being sought. The problem was first defined, for our purposes, the 1600s by Blaise Pascal and de Pierre de Fermat which launched modern probability theory (Devlin, 2008). It can be stated succinctly as:

"What decision should a person (customer) make when facing a decision involving risk?"

Centuries later Frank Knight drew a distinction between Risk and Uncertainty (Knight, 1921). A situation involving risk is where a range of outcomes exist and are known with each having an associated probability. For our purposes Knight's notion of risk will be employed. The stated problem lies in the field of risk not uncertainty. The intellectuals of the day maid rapid progress and by 1738 Daniel Bernoulli summarised the solution to the problem as (Bernoulli, 1954):

> "Expected values are computed by multiplying each possible gain (sic) [outcome] by the number of ways (sic) [times] in which it [the outcome] can occur, and then dividing the sum of these products by the total number of possible cases (times) ,,, "

So if there are C_i possible [outcomes] and the game is played N times and n_i of these are associated with C_i then the expected value of these N games is derived as follows:

The sum of N games, as mentioned by Bernoulli, is

$S_N = n_1 . C_1 + n_2 . C_2 \ldots \; n_N . C_N$ (1)

or

$S_N = \Sigma n_i . C_i$ (2)

And

$N = \Sigma n_i$ (3)

And the expected value is merely the arithmetical average.

$$\overline{S_N} = \frac{S_N}{N} = \mu \qquad (4)$$

Bernoulli indicated how to determine the expected valued but he did not explain what the relationship is between the expected value and the solution to the problem. What decision should a person (customer) make when facing a game of risk. The decision for which an answer is required is; what price should he pay to play the game of chance? He is to gamble an amount G (in Rands) per game; what value should that G be? Simple accounting helps to arrive at the solution and hence the relationship between the expected value and the solution. Putting this within the supply and demand framework; The supplier supplies the games and the gambler gambles. Assume the casino to be the supplier and the gambler plays N games; what amount should he gamble per game?

Well the supplier, the casino wants to make an accounting profit as does the gambler.

	Income	Expenditure	Profits	
Suppler	N.G	S_N	$(N.G - S_N)$	(6)
Gambler	Sn	N.G	$(S_N – N.G)$	(7)

From which it is clear the only situation acceptable to both the supplier and the gambler would be where the profit to both equals zero or $G = S_N/N$; the average or the expected value. The expected value is thus the breakeven value. The situation where neither the supplier (casio) not the gambler make a profit.

Insight into some issues which have come from economists over the centuries, is made at this point. It is these insights the visitor to the island of economics should take into consideration.

First determining the correct price to pay for the gamble requires that the expected value be determined. That requires knowledge of probability theory and it is unrealistic to believe every gambler can or will or will want to or does these calculations at arriving at the value of G. It is accepted that not every gambler is capable of doing probability calculations. G is not arrived at by the gambler from mere observation or intuition. The assumption is, if it is calculated, it has been correctly calculated by someone skilled in probability theory or from accounting records. It cannot be determined intuitively. The very reason probability theory was put forward in the first place was to have a theory to solve games of chance.

Second, not only must the gambler do these calculations but so must the supplier (casio). The casino can also arrive at the correct answer from its accounting records. If it accepts the incorrect value it will soon know; either the will make significant profits or losses. It will then adjust G until it arrives at the correct figure. If one casino operator can arrive at the correct answer then can all casino operators arrive at that figure, as can the regulator, a point articulated by Knight. Because the casino knows the correct price, it is not the gambler who sets the price. The price is set by the casino not the gambler which will be above a level to prevent the casio from experiencing a loss and competition prevents it from making abnormal profits.

Third for the accounting records to correctly reflect the price, N (equations (1)-(4)) the number of games played must be substantial. To determine the price a substantial number of games must be played but at least one party, the suppler or the gambler. Determining the price involves repetitive games. By price is meant the actual price paid not a calculation of what it is thought the price should be. It can be that the supplier is involved in a large number of games which individual consumers in a small number of games. In the case of a Lotto for example the Lotto operator may sell millions of tickets be game but in the lifetime the purchaser purchases, relatively speaking a small number of tickets.

Fourth should the process ensure the price is set at zero economic profits, there is no need for the gambler to hire his own mathematics expert. The market price is the zero economic profit price; it is the expected value. The market provides what Alfred Marshall called the consumer's surplus (Marshall, 1920). He pointed out that if customers had to set the price they will price it above the market value. The market gives the consumer the lower price, a surplus over the consumer's price; hence the term consumer's surplus. It is this realisation, which lead Knight to conclude that risk situations lead to zero profits. Profits are to be found in situations of uncertainty, not risk (Knight, 1921). The ability of the consumer to price the gamble plays a very small roll in the story.

Fifth, conceptually the expected value may point to zero profits but average is not a single point, it is described by a distribution something known for a long time (Feller, 1945). This fact rarely features in discussions involving decision making under conditions of risk.. Thus if the casino operator wants to be in a zero loss situation it needs to add a bit to the price G to cater for the distributional reality. The amount added is determined by the degree of certainty the casino operator can tolerate. Sometime later the Central Limit Theorem and Law of Large Numbers were uncovered making these matters clearer. So if the casino operator wants greater confidence he would need to charge:

$$G' = \mu + \sigma \qquad (8)$$

Where σ is chosen to give the casino operator the desired degree of confidence and it the normal distribution is assumed as a natural consequence of the Law of Large Numbers if .

Six, it is clear in reality a casino has expenses and if it is a business the purposes is to make profit: All of this is covered by the actual price paid, P, to gamble.

$$P = (\mu + \sigma) + \text{Expenses} + \text{RoI} \qquad (9)$$

This amount is well in excess of μ the expected value. The gambler will never in reality get his zero cost game. The cost of the gamble is always greater the expected financial benefit derived from the game. Nevertheless gambling exists. Clearly to the gambler the Utility of the gamble is greater than the Utility of retaining the price of the game,

$$U((\mu + \sigma) + \text{Expenses} + \text{RoI}) > U(P) \qquad (10)$$

Nicholas Bernoulli's red herring: The St Petersburg Game

The above is clear and simple so what is the issue? In 1713 Nicholas Bernoulli threw a spanner in the works. As it turns out a red herring. He thought he had devised a number of games which demonstrated the above theory is flawed. His correspondence has survived and is readily available (Pulskamp, 1999) He circulated these problems round Europe for a couple of years but no-could provide an answer to his queries. Eventually 25 years later a relative Daniel Bernoulli provide a non-solution to his problem which after being rediscovered by von Neumann and Morgenstein has played a central role in modern economics (von Neumann & Morgenstern, 1944). One of these problems has survived to this day and has been influential. This game has been dubbed the St Petersburg Game. The game is simple; flip a coin until a head appears and if it appears at the nth flip the payout is $2^{(n-1)}$. The associated probability is $1/2^n$.

It is not particularly difficult to determine the expected value game using equations (1) (2) and (3) above. The point of departure is to determine the number of games which terminate at each point, n_i in the series. If 2^k games are played then $2^{(k-1)}$ games are expected to terminate after the first throw; $2^{(k-2)}$ after the second throw and so on until the kth term were $2^{(k-k)}$ games are expected to terminate. The number of games at each termination points is thus"

$$2^0 + 2^1 + 2^2 + {}^{2k-1} \qquad (9)$$

The series is thus k in length with each term contributing to the expected value.

The i^{th} term in the series contributes:

$$1/2^i \;.\; 2^k/2^k \;.\; 2^{(i-1)} = 1/2 \qquad (10)$$

Each term in the series contributes ½ giving the sum of the k terms as k/2 (11)

However if the series in equation (9) is summed it will be seen the sum is:

$2^k - 1$ (12)

One game is expected to go beyond the series k in length. If it at the first flip beyond k it will contribute 1 to the expected value determined or 2 if it is at the second flip beyond k or 4 if it terminates at third flip and so on. So the expected value at a 50 percent confidence level is k/2 + 1 or k/2 + 2 at a 75 percent confidence level and so on.

So if a casino operator offers to gamblers to play the game say 2^14 times (16 384) the casino should charge an amount of R8 per game. Nicholas Bernoulli's faulty mathematics led him to believe the expected value of the St Petersburg game to be infinite. So in his view if a casino is to offer the game it should ask say R10 000 000 per game (and even this compared to infinity that is a small amount). Nicholas did not believe anyone would take up the offer and would offer no more than R20 to play the game.

Daniel Bernoulli's non-solution

So the intellectuals of the day faced a problem they could not solve. The expected value theory, according to their faulty mathematics, provided an answer to a problem involving risk, an infinite expected value, but common sense told them this was not going to be taken up. Faced with this paradox Daniel Bernoulli offered a different solution.

The coefficients of the series are: $1/2^i \,.\, 2^{(i-1)} = ½$. This is because the denominator (probability) and the numerator (outcome) increase at exactly the same rate. There was nothing he could do about the probability but the outcome represented wealth to be gained. He argued that if it is accepted that individuals have a marginal decreasing appreciation of wealth then the numerator would increase at a slower rate than the denominator. He used the log function as a proxy for the decreasing appreciation of wealth; marginal decreasing utility of wealth. He multiplied this utility by the probability and arrived at the expected utility value of the game. which was dependent on the initial wealth of the gambler. The wealthier the gambler the more the gambler would be prepared to risk. His calculations indicated if the gambler processed no wealth he would be prepared to gamble R2, if he owned R10 he would gamble R3; if he owned R100 he would gamble 4; If he owned1 000 he would be prepared to gamble R6. He would have to be very wealth to gamble R20.

What Daniel Bernoulli managed to do was to arrive at a more reasonable amount more and more in line with what it was generally believed rational gamblers would be willing to pay to play the game. He found a mechanism which produced an answer closer to what it was perceived persons would be willing to pay. The solution did not come from the mathematical determination of the game but a perception of the behaviour of gamblers. It can be said that Daniel Bernoulli is the father of behavioural economics.

Nevertheless Daniel Bernoulli's solution is a non-solution. The correct solution is one which produces a zero profit for both the casino and the gambler. The amount to gamble, in the case of risk can be objectively determined by the use of probability theory. In Daniel Bernoulli's utility solution probability theory is still used but it does not arrive at an objective solution and finally the amount to gamble is in fact objectively independent of wealth; the utility solution is dependent on wealth. So in his multibillionaire would be prepared to gamble say R20 per game and more. The accounting outcome of playing the game 2^{14} (16 384) times. The expected value is 9 at a confidence level of 74 percent and the series is expected to be 15 terms in length. The results are indicated below where the gambler stakes R2, R10 and R20 per game. If the expected value is R9 per game clearly the casino will not accept the wager of R2 per game.

Table 2: Playing the game 16 384 times; Expected value 9: 75 % confidence

	Income	Expenditure	Profit	Length of the series	Average payout
Casino (R2)	R32 768	R202 940	-R170 172	17	-10.386
Gambler (R2)	R202 940	R32 768	R170 172	17	-10.386
			R0		
Casino (R10)	R163 840	R118 327	R45 513	14	7.222
Gambler (R10	R118 327	R163 840	-R45 513	14	7.222
Casino (R20)	R163 840	R124 242	R39 598	14	7.583
Gambler (R20	R124 242	R163 840	-R39 598	14	7.583

As expected accepting R2 per game causes the casino to experience a loss of R170 172. The payout per game of R10.386 was larger than R9 at a 75 % confidence level. Betting amounts in excess of the expected value of R9 would result in a loss to the gambler as happened. Betting the large amount as calculated in 1738 of say R 1000 per game would ruin any gambler. The expected utility solution is clearly an improvement but still wrong. It is a non-solution but produces a considerably better outcome than the infinite expected value.

Daniel Kahanman's criticism of Daniel Bernoulli's solution

Daniel Kahneman, a psychologist was awared the Noble Prize in economics and no doubt has made a significant contribution in the field in economics. He is however very critical of Daniel Bernoulli solution of the St Petersburg Paradox. In the context of Bernoulli's actual article it is difficult to understand Kahneman's actual criticism. Indeed it is difficult to understand much of his criticism in general. Being a psychologist he sees the world in the context of individuals making decisions. As indicated above at least since Marshall introduced the notion of consumer's surplus it has been realised in the vast majority of cases where competitive markets exist the market price is set below that which individual's would be willing to pay. When it comes to price, the estimation of either the price or probabilities play very much a secondary roll. The market price is set as a consequence of competition. Economists have long since recognised that price plays a secondary role in entering into contracts. Utility plays the major roll. As Adam Smith pointed out (Smith, 1776)

Price = Benefit + Expenses + RoI

The price paid in a contract is almost always less than the benefit; nevertheless contracts are entered into. The economist explanation is

U(Contract) > U(P)

The utility of the contract exceeds the utility of retaining the price paid to enter into the contract.

Although ostensibily criticising Bernoulli's solution to the St Petersburg paradox he does not discuss Bernoulli's solution or the problem being addressed. How to price a gaime involving risk. In the context of the problem Bernoulli was trying address he does not provide any insight. He should have first understood the problem itself.

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